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Non-Lagrangian collective variable approach for optical solitons in fibres

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Abstract

We use a non-Lagrangian method to express the generalized nonlinear Schrödinger equation, for pulse propagation in optical fibres, in terms of the pulse parameters, called collective variables, such as the pulse *width*, *amplitude*, *chirp* and *frequency*. The collective variable equations of motion, which include the important effects due to fibre losses, third-order dispersion, stimulated Raman scattering and self-steepening are derived using a direct averaging method without the help of any Lagrangian.

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Intense light pulses propagating in optical fibres may induce a host of nonlinear phenomena such like parametric wave mixing, stimulated Raman scattering, or self-steepening [1, 2]. The combined effects of those nonlinear phenomena and the fibre chromatic dispersion, lead, in general, to complicated dynamical processes, which are particularly difficult to understand from a direct analysis of the original electromagnetic field associated with the light pulse, say ϕ. To easily understand such dynamical processes, one generally attempts to reduce the dynamics of the pulse field ϕ, which involves an infinite (in practice, a very large) number of degrees of freedom, to the dynamics of a simple mechanical system having only a few degrees of freedom. The corresponding mechanical system may take the form of a particle embedded in a substrate potential, a diatomic molecule, or even a more complicated molecular system, depending on the degree of complexity of the pulse dynamics. Then, one associates each degree of freedom of the mechanical system with a variable, called a *collective variable* (CV), describing a relevant physical parameter for the pulse (temporal position, amplitude, width, etc). The achievement of this CV approach depends on the possibility of transforming the partial differential equation for the original field ϕ, into a set of ordinary differential equations for the CVs. Although such CV approaches have been applied successfully to condensed matter systems, in particular to nonlinear Klein–Gordon and similar systems [3–7], the actual stage of CV treatments of nonlinear partial differential equations in nonlinear fibre optics

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happens to be surprisingly much less elaborated than in condensed matter physics. The reason for this is that, from the period of invention of optical solitons up to very recent years, the main line of research in ultra-high-capacity fibre communications was based on the concept of the 'classical soliton', which represents an exact balance between the fibre group-velocity dispersion and its intensity-dependent refractive index. This soliton arises as a solution of the standard nonlinear Schrödinger equation (NLSE), with a well known hyperbolic secant profile that can be obtained by different techniques (e.g. the inverse scattering transform [8]), without going through any CV approach [9].

One of the major lines of current research in optics communications focuses on the modelling of soliton transmissions in dispersion-managed (DM) fibre-optic links, with a view both to the upgrade of the capacity of existing terrestrial networks, and the design of submarine fibre systems [10]. Basically, the dispersion-management technique consists of using a transmission line with a periodic dispersion map such that each period is built up by two types of fibres of generally different lengths and opposite group-velocity dispersion [10]. It is now well recognized that the DM technique is one of the most promising way to achieve ultra-high-bit rate transmissions (multiterabit) over thousands of km in near-future systems [10]. The main limitations in the performance of such fibre links come from detrimental linear and nonlinear effects; such as the cross-phase modulation, filtering, phase and amplitude modulation, third-order dispersion, stimulated-Raman scattering effects, or self-steepening. The dynamical equation that governs the nonlinear pulse propagation in DM fibres with all higher-order effects, is the generalized nonlinear Schrödinger equation [10]. Because of the functional dependence of the fibre parameters upon the propagation coordinate, the NLSE is not at all integrable. In this context, the CV approach is very much useful for the study of the dynamical behaviour of the optical pulses in DM fibres.

In most of the early literature that has used CVs in the context of optical solitons in fibres, a significant effort was made to analyse the soliton dynamics under the influence of various perturbing factors such as third-order dispersion, stimulated Raman scattering or self-steepening [11–17]. Different perturbation theories were developed, using the adiabatic variation of conserved quantities of the NLSE [11], the adiabatic variation of the scattering data based on the inverse scattering transform [12–14], or the Lagrangian perturbation theory [15]. However, these perturbation theories yield consistent results only in the limit of weak perturbations. This limitation has led to the formulation of non-perturbative CV theories for the NLSE using the averaged Lagrangian method [10, 18–20]. Most of the recent and current theoretical developments have employed this Lagrangian method (also known as the *variational approach*) to describe pulse propagation in DM optical fibre transmission lines [19, 20].

The Lagrangian method basically needs the exact form of the Lagrangian corresponding to the NLSE to derive the CV equations of motion. If we consider the effects of optical losses, higher-order dispersion, stimulated Raman scattering and self-steepening, the dynamics of the pulse propagation in optical fibres is governed by a generalized NLSE [1, 2]. A Lagrangian corresponding to the generalized NLSE is not available. So it is not possible to derive the CV equations for the generalized NLSE using the Lagrangian method without any perturbation theory. In this paper, we derive the explicit form of the CV equations for the generalized NLSE using a direct averaging approach, which does not require any Lagrangian. We use a direct averaging method to derive the CV equations for nonlinear Schrödinger solitons with the help of the Ehrenfest theorem [21–24].

The NLSE takes the form [1, 2]

$$\dot{\psi} + \mathrm{i}\frac{\beta_2}{2}\psi_{tt} - \mathrm{i}\gamma|\psi|^2\psi = 0 \tag{1}$$

where ψ is the slowly varying envelope of the axial field, overdot and subscript t denote the spatial and temporal partial derivatives, respectively. β_2 and γ represent the group-velocity dispersion and self-phase modulation parameters, respectively.

As a first step let us express the original field ψ in the coupled amplitude–phase form as

$$\psi(z,t) = \phi(z,t)\exp[\mathrm{i}S(z,t)] \tag{2}$$

where ϕ and S represent the amplitude and phase of ψ, respectively.

By substituting equation (2) into equation (1), we obtain the equations for amplitude and phase, respectively, from real and imaginary parts as

$$\frac{\dot{\phi}}{\phi} = \frac{\beta_2\phi_t S_t}{\phi} + \frac{\beta_2 S_{tt}}{2} \tag{3a}$$

$$\dot{S} = \gamma\phi^2 + \frac{\beta_2 S_t^2}{2} - \frac{\beta_2\phi_{tt}}{2\phi}. \tag{3b}$$

With the new variables defined as $v \equiv S_t$, $a(z,t) \equiv \phi_t/\phi$ and $h(z,t) \equiv -\dot{\phi}/[\beta_2\phi]$, we rewrite equation (3$a$) as

$$\frac{\partial v}{\partial t} + 2a(z,t)v + 2h(z,t) = 0. \tag{4}$$

Integrating equation (4) with respect to time yields

$$v(z,t) = \frac{v_0(z,t)}{\phi^2} \tag{5}$$

where v_0 is determined by

$$\frac{\partial v_0(z,t)}{\partial t} = \frac{2\phi\dot{\phi}}{\beta_2}. \tag{6}$$

At this point, we assume an ansatz function for ϕ in the form

$$\phi(z,t) = X_1(z)\tilde{\phi}\left[\frac{t - X_2(z)}{X_3(z)}\right] \tag{7}$$

which introduces three collective variables $X_1(z)$, $X_2(z)$ and $X_3(z)$ into the system, where $X_1(z)$ represents the amplitude, $X_2(z) \equiv \langle t\rangle$ describes the temporal position and $X_3(z)$ is the width of the pulse. Here the averaging is the usual quantum expectation value

$$\langle\hat{O}\rangle = \int_{-\infty}^{\infty} \psi^*(z,t)\hat{O}\psi(z,t)\,\mathrm{d}t. \tag{8}$$

Substituting the ansatz (7) into equation (6) and integrating the resulting equation yields

$$v_0(z,t) = \frac{-\dot{X}_3}{\beta_2}\frac{[t - X_2]}{X_3}\phi^2 - \frac{\dot{X}_2}{\beta_2}\phi^2 + R(z) + \frac{1}{2X_3}\frac{\mathrm{d}(X_3X_1^2)}{\mathrm{d}z}\int\tilde{\phi}^2\,\mathrm{d}t. \tag{9}$$

Then, equation (5) becomes

$$v(z,t) = \frac{-\dot{X}_3}{\beta_2}\frac{[t - X_2]}{X_3} - \frac{\dot{X}_2}{\beta_2} + \frac{R(z)}{\phi^2} + \frac{1}{2\phi^2X_3}\frac{\mathrm{d}(X_3X_1^2)}{\mathrm{d}z}\int\tilde{\phi}^2\,\mathrm{d}t \tag{10}$$

where $R(z)$ is a constant of integration.

The last term on the right-hand side of equation (10) contains the term $\int\tilde{\phi}^2\,\mathrm{d}t$ which must be completely worked out to explicitly express v, and then S, in terms of the collective variables. To continue the procedure of derivation of the CV equations of motion, we consider a class of ansatz satisfying

$$\frac{\mathrm{d}}{\mathrm{d}z}(X_3 X_1^2) = 0 \tag{11}$$

which cancels the last term on the right-hand side of equation (10). Note that equation (11) implies that

$$E_0 \propto X_3(z) X_1^2(z) = X_3(0) X_1^2(0). \tag{12}$$

In other words, using equations (11) and (12), we assume, in fact, that the energy of the soliton (E_0) is conserved during the dynamics. In this situation, the ansatz in equation (7) becomes

$$\phi(z,t) = \sqrt{\frac{E_0}{X_3}}\tilde{\phi}\left[\frac{t - X_2(z)}{X_3(z)}\right]. \tag{13}$$

At this point, we define a variable, $\xi \equiv [t - X_2(z)]/X_3(z)$ and thus $\tilde{\phi}$ will become an explicit function of ξ. On the other hand, in equation (10), the constant of integration $R(z)$ must be set to zero to maintain finite variations of S_t for all t, and in particular, for soliton boundary conditions: $\phi(t = \pm\infty) = 0$. Then, the integration of equation (10) with respect to time yields explicitly the functional of the phase upon the collective variables X_2 and X_3,

$$S(z,t) = \frac{-\dot{X}_3(z)}{2\beta_2 X_3(z)}[t - X_2(z)]^2 - \frac{\dot{X}_2}{\beta_2}[t - X_2(z)] + X_6(z) \tag{14}$$

where $X_6(z)$ is a constant of integration which represents the phase of the pulse.

Here, it should be once again emphasized that one of the main goals of our collective variable approach is to show that CV equations can be obtained without having to explicitly construct the Lagrangian associated with the original NLSE. We only need to know the functional dependence of the amplitude ϕ and phase S upon the collective variables. Then, all the CV equations can be directly obtained from the original field equations (3*a*) and (3*b*), through an iterative procedure of differentiation and subsequent averaging. To be more explicit, substitution of equation (14) into equation (3*b*) yields

$$-\frac{X_3}{\beta_2}\ddot{X}_2\xi - \frac{\ddot{X}_3 X_3}{2\beta_2}\xi^2 + G(z) = f \tag{15}$$

where

$$f = -\frac{\gamma X_1^2}{X_3}\tilde{\phi}^2 - \frac{\beta_2}{2X_3^2}\frac{\tilde{\phi}_{\xi\xi}}{\tilde{\phi}} \tag{16a}$$

$$G(z) = \dot{X}_6 + \frac{\dot{X}_2^2}{2\beta_2}. \tag{16b}$$

To obtain the equations for $\ddot{X}_2$, $\ddot{X}_3$ and G, we proceed as follows.

Averaging equation (15) yields

$$-\frac{\ddot{X}_3 X_3}{2\beta_2}\eta^2 + N_0 G(z) = -\gamma\frac{X_1^2}{X_3}\delta^2 + \frac{\beta_2\mu^2}{2X_3^2} \tag{17}$$

where $\mu^2 = \int_{-\infty}^{\infty}\tilde{\phi}_\xi^2\,\mathrm{d}\xi$, $\delta^2 = \int_{-\infty}^{\infty}\tilde{\phi}^4\,\mathrm{d}\xi$, $\eta^2 = \int_{-\infty}^{\infty}\xi^2\tilde{\phi}^2\,\mathrm{d}\xi$ and $N_0 = \int_{-\infty}^{\infty}\tilde{\phi}^2\,\mathrm{d}\xi$.

To obtain the next equation there are two options: one can either multiply equation (15) by ξ or differentiate equation (15) with respect to ξ. We follow the later to obtain

$$-\frac{X_3}{\beta_2}\ddot{X}_2 - \frac{\ddot{X}_3 X_3}{\beta_2}\xi = \frac{\mathrm{d}f}{\mathrm{d}\xi}. \tag{18}$$

Averaging equation (18) yields

$$\ddot{X}_2 = 0. \tag{19}$$

An important point to be noticed here is that averaging equation (18) does not require the explicit calculation of the corresponding integrals. Indeed, the term multiplying $\ddot{X}_3$ in equation (18) is an odd function of ξ (since f is an even function of ξ); its average is zero.

To obtain the third equation, which will determine together with equation (17) and (19) the whole set of functions $\ddot{X}_2$, $\ddot{X}_3$ and G, one can either multiply equation (18) by ξ or differentiate with respect to ξ. However, an interesting point to be emphasized here is that these two options are not strictly equivalent in the sense that they do not formally lead to the same equation. Indeed, in general, one of these two options will generate many coefficients that differ from those already generated (μ^2, δ^2, η^2 and N_0), thus increasing the number of calculations. Consequently, a general rule which one can formulate is to systematically choose the option which leads to no or fewer additional coefficients. Thus multiplying both sides of equation (18) by ξ and taking the average of the resulting equation yields

$$\ddot{X}_3 = \frac{\beta_2^2\mu^2}{X_3^3\eta^2} - \frac{\gamma\beta_2\delta^2X_1^2}{2X_3^2\eta^2}. \tag{20}$$

Finally, equation (15) determines the profile of the function $\tilde{\phi}$ as

$$\frac{\mathrm{d}^2\tilde{\phi}}{\mathrm{d}\xi^2} + \frac{2\gamma X_3X_1^2}{\beta_2}\tilde{\phi}^3 + \frac{2X_3^2}{\beta_2}\left[G(z) - \frac{X_3\ddot{X}_3}{2\beta_2}\xi^2\right]\tilde{\phi} = 0 \tag{21}$$

where $\tilde{\phi}(0) = 1$, and

$$G(z) = \frac{1}{X_3N_0}\left[\beta_2\frac{\mu^2}{X_3} - \frac{5}{4}\gamma X_1^2\delta^2\right]. \tag{22}$$

In all the previous calculations, pulse parameters are described by equations which are like Newton equations of motion for a particle in a given potential. It is also possible and useful to express the CV equations of motion in terms of first-order derivative of CVs with respect to z. Here we assume the following form for $\tilde{\phi}$ and S:

$$\tilde{\phi} = \exp\left[\frac{-(t-X_2)^2}{X_3^2}\right] \tag{23a}$$

$$S = \tfrac{1}{2}X_4(t-X_2)^2 + X_5(t-X_2) + X_6 \tag{23b}$$

where $X_4(z)$ and $X_5(z)$ represent the pulse chirp and central frequency, respectively. Then comparing equations (14) and (23*b*) we find

$$\dot{X}_3 = -\beta_2X_4X_3 \tag{24a}$$

$$\dot{X}_2 = -\beta_2X_5. \tag{24b}$$

Using equations (11), (16*b*), (19), (20) and (22) we obtain the remaining CV equations of motion as

$$\dot{X}_1 = \frac{\beta_2}{2}X_1X_4 \tag{25a}$$

$$\dot{X}_4 = -\beta_2\left(\frac{4}{X_3^4} - X_4^2\right) - \frac{\sqrt{2}\gamma X_1^2}{X_3^2} \tag{25b}$$

$$\dot{X}_5 = 0 \tag{25c}$$

$$\dot{X}_6 = -\beta_2\left(\frac{X_5^2}{2} - \frac{1}{X_3^2}\right) + \frac{5}{4\sqrt{2}}\gamma X_1^2. \tag{25d}$$

Thus, we have derived the CV equations of motion for the NLSE (1) using the direct averaging method without the help of the Lagrangian. One can also derive the same CV equations using the variational approach from the Lagrangian corresponding to the NLSE (1). As the direct averaging method does not require a Lagrangian for the derivation of the CV equations, in the following we derive the CV equations for a dissipative system such as the generalized NLSE.

The generalized NLSE takes the form [1, 2]

$$\dot{\psi} + \mathrm{i}\frac{\beta_2}{2}\psi_{tt} - \mathrm{i}\gamma|\psi|^2\psi = -\frac{\alpha}{2}\psi + \frac{\beta_3}{6}\psi_{ttt} - \mathrm{i}\psi\gamma_r(|\psi|^2)_t - \gamma_s(|\psi|^2\psi)_t \quad (26)$$

where α, β_3, γ_r and γ_s represent the fibre losses, third-order dispersion, stimulated Raman scattering and self-steepening parameters, respectively.

Substituting equation (2) into (26) leads to the following set of coupled equations:

$$\frac{\dot{\phi}}{\phi} = \frac{\beta_2\phi_t S_t}{\phi} + \frac{\beta_2}{2}S_{tt} - \frac{\alpha}{2} + \frac{\beta_3}{6}\left[\frac{\phi_{ttt}}{\phi} - 3\frac{\phi_t S_t^2}{\phi} - 3S_t S_{tt}\right] - 3\gamma_s\phi\phi_t \quad (27a)$$

$$\dot{S} = \gamma\phi^2 + \frac{\beta_2 S_t^2}{2} - \frac{\beta_2\phi_{tt}}{2\phi} + \frac{\beta_3}{6}\left[\frac{3\phi_{tt}S_t}{\phi} + \frac{3\phi_t S_{tt}}{\phi} + S_{ttt} - S_t^3\right] - \gamma_s S_t\phi^2 - 2\gamma_r\phi\phi_t. \quad (27b)$$

Solving equations (27) directly for $S(z,t)$ (as we did earlier for NLSE) is extremely complicated. In such a situation, straight away one has to postulate a suitable ansatz for ϕ and S, and thus, the precise form of the ansatz functions which introduce the CVs become crucial. Equations (7) and (23*b*) show that ansatz functions for the NLSE (1) are given by

$$\phi(z,t) = X_1(z)\tilde{\phi}\left[\frac{t - X_2(z)}{X_3(z)}\right] \quad (28a)$$

$$S(z,t) = \tfrac{1}{2}X_4(z)[t - X_2(z)]^2 + X_5(z)[t - X_2(z)] + X_6(z) \quad (28b)$$

which contain six collective variables, X_1, X_2, X_3, X_4, X_5 and X_6, which have a proper physical meaning. Also, for generalized NLSE (26) also, we take the same ansatz functions, but for simplicity and without loss of generality, we assume $\tilde{\phi}$ to be a Gaussian. One can also assume any other desired form for the ansatz functions which give a proper physical meaning for the CVs. Substituting the ansatz functions (28) into equations (27), we obtain

$$\begin{aligned}
-(X_5 + X_3X_4\xi)\dot{X}_2 + \frac{X_3^2\dot{X}_4\xi^2}{2} + X_3\dot{X}_5\xi + \dot{X}_6 &= \beta_2\left(-\frac{2}{X_3^2} + \frac{X_3^2X_4^2}{2}\right)\xi^2 + \beta_2X_3X_4X_5\xi \\
&+\beta_2\left(\frac{1}{X_3^2} + \frac{X_5^2}{2}\right) + \gamma X_1^2\tilde{\phi}^2 + \beta_3\left(\frac{2X_4}{X_3} - \frac{X_3^3X_4^3}{6}\right)\xi^3 \\
&+\beta_3\left(\frac{2X_5}{X_3^2} - \frac{X_3^2X_4^2X_5}{2}\right)\xi^2 - \beta_3\left(\frac{2X_4}{X_3} + \frac{X_3X_4X_5^2}{2}\right)\xi \\
&-\beta_3\left(\frac{X_5}{X_3^2} + \frac{X_5^3}{6}\right) + \frac{4\gamma_r X_1^2\xi\tilde{\phi}^2}{X_3} - \gamma_s(X_1^2X_5 + X_1^2X_3X_4\xi)\tilde{\phi}^2 \quad (29a)
\end{aligned}$$

$$\begin{aligned}
\frac{\dot{X}_1}{X_1} + \frac{\dot{X}_3\xi^2}{2X_3} + \frac{\dot{X}_2\xi}{X_3} &= -2\beta_2X_4\xi^2 - \frac{2\beta_2X_5\xi}{X_3} + \frac{\beta_2X_4}{2} - \frac{\alpha X_1}{2} - \frac{\beta_3X_4X_5}{2} + \frac{6\gamma_sX_1^2\xi\tilde{\phi}^2}{X_3} \\
&+\beta_3\left(X_3X_4^2 - \frac{4}{3X_3^3}\right)\xi^3 + 2\beta_3X_4X_5\xi^2 + \beta_3\left(\frac{2}{X_3^3} - \frac{X_3X_4^2}{2} + \frac{X_5^2}{X_3}\right)\xi. \quad (29b)
\end{aligned}$$

Now, we apply the direct averaging procedure to obtain the CV equations of motion. We present the calculations in three steps as follows:

Step 1. Taking the average of equations (29*a*) and (29*b*) yields

$$-X_5\dot{X}_2 + \frac{X_3^2}{8}\dot{X}_4 + \dot{X}_6 = \frac{\beta_2}{2}\left(\frac{1}{X_3^2} + \frac{X_3^2X_4^2}{4} + X_5^2\right) + \frac{\gamma X_1^2}{\sqrt{2}} - \frac{\beta_3}{2}\left(\frac{X_5}{X_3^2} + \frac{X_3^2X_4^2X_5}{4} + \frac{X_5^3}{3}\right) - \frac{\gamma_s X_1^2 X_5}{\sqrt{2}} \tag{30a}$$

$$\frac{\dot{X}_1}{X_1} + \frac{\dot{X}_3}{2X_3} = -\frac{\alpha}{2}. \tag{30b}$$

Step 2. Multiplying equations (29*a*) and (29*b*) by ξ and averaging the resulting equation yields

$$\dot{X}_5 - X_4\dot{X}_2 = \beta_2 X_4 X_5 - \beta_3\left(\frac{X_4}{2X_3^2} + \frac{X_4X_5^2}{2} + \frac{X_3^2X_4^3}{8}\right) + \frac{\sqrt{2}\gamma_r X_1^2}{X_3^2} - \frac{\gamma_s X_1^2 X_4}{2\sqrt{2}} \tag{31a}$$

$$\dot{X}_2 = -\beta_2 X_5 + \beta_3\left(\frac{1}{2X_3^2} + \frac{X_5^2}{2} + \frac{X_3^2X_4^2}{8}\right) + \frac{3\gamma_s X_1^2}{2\sqrt{2}}. \tag{31b}$$

Step 3. Differentiating equations (29*a*) and (29*b*) twice, with respect to ξ and averaging the resulting equation yields

$$\dot{X}_4 = -\beta_2\left(\frac{4}{X_3^4} - X_4^2\right) - \frac{\sqrt{2}\gamma X_1^2}{X_3^2} + \beta_3\left(\frac{4X_5}{X_3^4} - X_4^2X_5\right) + \frac{\sqrt{2}\gamma_s X_1^2 X_5}{X_3^2} \tag{32a}$$

$$\dot{X}_3 = -\beta_2 X_3 X_4 + \beta_3 X_3 X_4 X_5. \tag{32b}$$

Thus, equations (30)–(32) give the simplest system of six equations for the six CV equations of motion. Solving the system of equations (30)–(32), we obtain the following explicit form for the CV equations of motion:

$$\dot{X}_1 = -\frac{\alpha}{2}X_1 + \frac{\beta_2}{2}X_1X_4 - \frac{\beta_3}{2}X_1X_4X_5 \tag{33a}$$

$$\dot{X}_2 = -\beta_2 X_5 + \beta_3\left(\frac{1}{2X_3^2} + \frac{X_5^2}{2} + \frac{X_3^2X_4^2}{8}\right) + \frac{3}{2\sqrt{2}}\gamma_s X_1^2 \tag{33b}$$

$$\dot{X}_3 = -\beta_2 X_3 X_4 + \beta_3 X_3 X_4 X_5 \tag{33c}$$

$$\dot{X}_4 = -\beta_2\left(\frac{4}{X_3^4} - X_4^2\right) - \frac{\sqrt{2}\gamma X_1^2}{X_3^2} + \beta_3\left(\frac{4X_5}{X_3^4} - X_4^2X_5\right) + \frac{\sqrt{2}\gamma_s X_1^2 X_5}{X_3^2} \tag{33d}$$

$$\dot{X}_5 = \frac{\sqrt{2}\gamma_r X_1^2}{X_3^2} + \frac{\gamma_s X_1^2 X_4}{\sqrt{2}} \tag{33e}$$

$$\dot{X}_6 = \beta_2\left(\frac{1}{X_3^2} - \frac{X_5^2}{2}\right) + \frac{5\gamma X_1^2}{4\sqrt{2}} + \beta_3\left(\frac{X_5^3}{3} + \frac{X_3^2X_4^2X_5}{8} - \frac{X_5}{2X_3^2}\right) + \frac{\gamma_s X_1^2 X_5}{4\sqrt{2}}. \tag{33f}$$

It is interesting to mention that in the absence of fibre losses, third-order dispersion, stimulated Raman scattering and self-steepening (i.e. $\alpha = \beta_3 = \gamma_r = \gamma_s = 0$), the CV equations (33) reduce to the CV equations (24) and (25).

To conclude, we have successfully derived the CV equations of motion for the generalized NLSE (26), which includes important higher-order effects for pulse propagation in optical fibres, without the help of any Lagrangian. Importantly, in nonlinear fibre optics, there exists a host of nonlinear partial differential equations, which do not possess the Lagrangian function. In such cases, one cannot apply the Lagrangian approach, whereas this direct averaging approach can be used in all cases for obtaining the CV equations of motion without the help of the Lagrangian function. With these CV equations, a deep insight into the modification of the dynamics due to a particular effect of any higher-order term of the generalized NLSE can be obtained. We have shown how to derive the CV equations of motion for both conservative (NLSE) and non-conservative (generalized NLSE) systems without any perturbation theory. Hence, in a similar way one can also derive the CV equations for any system equation with other effects due to the presence of periodic amplifiers, filters, etc, in optical fibre transmission lines.

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